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Supergravity Gaugings and Moduli Superpotentials

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Abstract

These lecture notes describe the method of $N = 4$ supergravity gaugings used as a four-dimensional effective Lagrangian description of the moduli superpotentials generated by superstring vacua with fluxes.

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1 Introduction

String compactifications on simple, symmetric backgrounds produce in general several or many massless scalar fields of geometric origin called *moduli*. The perception of their role in a description of particle physics from a fundamental superstring theory is ambivalent. On one side, they are welcome since we may expect to obtain new scales and parameters from their expectation (or background) values. This is useful since string theory has very few fundamental parameters and we need to find an origin for the quantities appearing as parameters of the standard model. Secondly, we need to *understand* these spontaneously generated scales and parameters to obtain a predictive theory and we also need to give masses to the moduli, to avoid phenomenologically unwanted massless or very light scalar fields. Generating in the superstring theory background values and masses of the moduli fields is the problem of *moduli stabilization*.

In a compactification to four dimensions with a residual $N = 1$ supersymmetry, as in models relevant to phenomenology, moduli can be stabilized if an appropriate superpotential is generated in the process. Also, since the mechanism of (low-energy) spontaneous supersymmety breaking is strongly affected by the presence and the structure of an effective superpotential, the problems of moduli stabilization and supersymmetry breaking are deeply related.

Various sources of superpotentials have been identified. Firstly, it has been recognized many years ago that background values of the three-form field present in the massless spectrum of heterotic strings lead to a superpotential [1, 2]. This *flux* of the three-form field is compatible with field equations and supersymmetry variations of the ten-dimensional theory. Simultaneously, it was found that this "perturbative" superpotential can be supplemented by a non-perturbative contribution generated by gaugino condensates [1, 3]. Scherk-Schwarz compactifications [4] produce in general moduli-dependent superpotentials. More recently, models based on type II orientifolds with their D–brane systems have been of primary interest also because of the richness of their NS–NS and R–R fluxes.

As in many physics problems where the fundamental theory is not sufficiently understood, the problem of moduli stabilization can be approached using (at least) two complementary methods. The first approach is to solve the string equations for specific backgrounds or classes of backgrounds. It provides a derivation of the effective superpotential for these classes of vacua. While rigourous and satisfactory, this method does not provide a general study of the problem. In addition, switching on fluxes often transforms a simple compactification into a hardly solvable problem of geometry. A vast literature has been devoted to studies of *compactifications with fluxes* in many classes of superstring backgrounds.[1] This approach of the moduli problem is not the subject of the present notes.

The second approach uses four-dimensional effective Lagrangians. The idea underlying the effective field theory method is to translate the known properties, and in particular the symmetry content, of the underlying fundamental theory into constraints on a field theory description of the light (four-dimensional) modes only. This effective field theory is then a tool to investigate various aspects of the expected low-energy physics predicted by the fundamental theory. It is also used to isolate the situations relevant to phenomenology and then the classes of vacua deserving a full-fledged, ten-dimensional study. The effective Lagrangian approach can be viewed as a *bottom-up* approach, in contrast to the *top-down* method provided by direct studies of compactifications with fluxes in classes of string vacua.

Superstring vacua relevant to phenomenology have sixteen supercharges. This large class of solutions includes in particular heterotic (and type I) strings and type II orientifolds. In four dimensions, sixteen supercharges lead to $N = 4$ supergravity coupled to the $N = 4$ super-Yang-Mills system. This theory has a severely restricted structure. The sigma-model defining its scalar sector is for instance unique. In fact, the only freedom to introduce parameters in $N = 4$ supergravity resides in the choice of *gauging* applied to its vector fields and multiplets. Hence, a gauged $N = 4$ supergravity treatment of the massless modes of superstring compactifications, supplemented with a breaking mechanism to $N = 1$ for potentially realistic compactifications, seems an appropriate starting point for an effective Lagrangian description of string compactifications: it is expected that the gauging parameters of the effective $N = 4$ supergravity theory encode the data of underlying string vacua, including non-trivial fluxes. This approach of $N = 4$ supergravity gaugings used for the derivation and study of moduli superpotentials has been developed in ref. [6][2], and expanded to the inclusion of non-perturbative gaugino condensates in ref. [8].

The purpose of the present notes[3] is to describe the method of supergravity gaugings in relation with the effective description of string moduli physics. They do not however discuss the application of the method to the study of specific physics problems or classes of compactifications with fluxes. The next four sections describe in general terms various aspects of field theory and supergravity gaugings, starting with elementary considerations. After a detailed discussion of the relevant aspects of $N = 4$ supergravity (section 6), the specific use of $N = 4$ supergravity gaugings to describe moduli effective supergravities is the subject of sections 7, which discusses in a simple orbifold the

[1] For a recent review and references, see [5].

[2] Expanding on a formalism used in an earlier study of finite-temperature superstring phases [7].

[3] Which follow from a lecture primary devised for PhD students and young postdocs.

reduction to $N = 1$ supergravity, and 8 where the identification of string moduli in terms of $N = 1$ superfields is studied.

2 Gauging, elementary facts

We begin with some very simple facts from gauge theory, slightly rephrased in view of the needs of the next sections.

Consider a Lie algebra with (real) structure constants $f_{AB}{}^{C} = -f_{BA}{}^{C}$. For any representation with generators T_A, the Lie algebra is

$$[T_A, T_B] = f_{AB}{}^{C} T_C \,. \tag{1}$$

With this convention, (finite-dimensional) representations of the algebra of a compact group have antihermitian generators. The Jacobi identity $[[T_A, T_B], T_C] + [[T_C, T_A], T_B] + [[T_B, T_C], T_A] = 0$ leads to

$$f_{AB}{}^{D} f_{DC}{}^{E} + f_{CA}{}^{D} f_{DB}{}^{E} + f_{BC}{}^{D} f_{DA}{}^{E} = 0, \tag{2}$$

which also implies

$$f_{AB}{}^{D} f_{DE}{}^{E} = 0. \tag{3}$$

A generic set of fields ϕ^j would infinitesimally transform according to

$$\delta\phi^j = \Lambda^A (T_A)^j{}_k \phi^k,$$

with infinitesimal parameters $\Lambda^A(x)$. Covariant derivatives use gauge fields $A_\mu = A_\mu^A T_A$ as connections:

$$D_\mu \phi^j = \partial_\mu \phi^j - A_\mu^A (T_A)^j{}_k \phi^k. \tag{4}$$

Their variation is

$$\delta A_\mu^A = \partial_\mu \Lambda^A + f_{BC}{}^{A} \Lambda^B A_\mu^C, \qquad\qquad \delta A_\mu = \partial_\mu \Lambda + [\Lambda, A_\mu]. \tag{5}$$

The generators of the adjoint representation can be defined as

$$(T_A)^B{}_C = f_{AC}{}^{B}. \tag{6}$$

Traceless generators, as in semi-simple algebras, automatically verify condition (3). If ϕ^j is in the adjoint representation (it is then ϕ^A),

$$\delta\phi^A = f_{BC}{}^{A} \Lambda^B \phi^C, \qquad\qquad \delta\phi = [\Lambda, \phi],$$

as in the second term in δA_μ^A. It follows that the *gauge curvatures*

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \\ F_{\mu\nu}^A &= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - f_{BC}{}^A A_\mu^B A_\nu^C \end{aligned} \tag{7}$$

transform in the adjoint representation,

$$\delta F_{\mu\nu} = [\Lambda, F_{\mu\nu}], \qquad\qquad \delta F_{\mu\nu}^A = f_{BC}{}^A \Lambda^B F_{\mu\nu}^C. \tag{8}$$

The propagation of physical gauge degrees of freedom is controlled by a Lagrangian quadratic in the curvatures:

$$\mathcal{L}_{kin.} = -\frac{1}{4} e \mathcal{N}_{AB}\, F_{\mu\nu}^A F^{B\,\mu\nu}, \tag{9}$$

where the *kinetic metric* $\mathcal{N}_{AB}$ is symmetric and non-degenerate to propagate all gauge fields.[4] Imposing gauge invariance, which ensures that two states with helicities ± 1 propagate for each vector field, leads to the equation

$$\delta\mathcal{L}_{kin.} = -\frac{1}{2} \mathcal{N}_{AB}\, F_{\mu\nu}^A \delta F^{B\,\mu\nu} = \frac{1}{2} F_{\mu\nu}^A F^{B\,\mu\nu} \Lambda^C \left(f_{CB}{}^D \mathcal{N}_{DA} \right) = 0$$

for arbitrary parameters Λ^C. The quantity in parentheses should then be antisymmetric under $A \leftrightarrow B$, which is the condition imposed on $\mathcal{N}_{AB}$ by gauge invariance:

$$f_{CA}{}^D \mathcal{N}_{DB} + f_{CB}{}^D \mathcal{N}_{DA} = 2 f_{C(A}{}^D \mathcal{N}_{B)D} = 0\,. \tag{10}$$

Defining

$$\hat{f}_{ABC} = f_{AB}{}^D \mathcal{N}_{DC} = -\hat{f}_{BAC}\,, \qquad\qquad f_{AB}{}^C = \hat{f}_{ABD}\, \mathcal{N}^{DC}, \tag{11}$$

where $\mathcal{N}^{AB}$ is the inverse of $\mathcal{N}_{AB}$ (which is nondegenerate), condition (10) requires full antisymmetry of the quantities $\hat{f}_{ABC}$, which are not structure constants of the gauge algebra since the kinetic metric is in general an arbitrary solution of eqs. (10) or (11).

There is a well-known particular solution. Consider a simple Lie algebra. For an arbitrary representation R, define

$$\mathrm{Tr}(T_A T_B) = -T(R)\, g_{AB} \tag{12}$$

where $T(R)$ is the Dynkin index of representation R and g_{AB} is the *Cartan metric*.[5] For a simple algebra, g_{AB} is non degenerate.[6] For the adjoint representation

$$T(Adj.)\, g_{AB} = -f_{AC}{}^D f_{BD}{}^C, \tag{13}$$

[4] In a quantum field theory, positivity of $\mathcal{N}_{AB}$ is required.

[5] There is a normalisation ambiguity in the generators and in the Dynkin indices.

[6] In our conventions, it is positive for compact semi-simple groups.

and $T(Adj.) = C(G)$, the quadratic Casimir number of the algebra G. Define then

$$f_{ABC} = f_{AB}{}^{D}\, g_{CD}. \tag{14}$$

The identity $\mathrm{Tr}([T_A, T_B]T_C) = \mathrm{Tr}([T_B, T_C]T_A) = \mathrm{Tr}([T_C, T_A]T_B)$ leads to

$$f_{AB}{}^{D} g_{DC} = f_{BC}{}^{D} g_{DA} = f_{CA}{}^{D} g_{DB},$$

and the structure constants f_{ABC}, as defined in eq. (14), are completely antisymmetric. The natural solution to condition (10) for a simple gauge group is then to identify

$$\mathcal{N}_{AB} = \frac{1}{g^2}\, g_{AB}. \tag{15}$$

Choosing g_{AB} to have eigenvalues ± 1, the real number g is the gauge coupling constant of the theory. The extension of this solution to a semi-simple algebra is obvious: there will be one arbitrary gauge coupling constant for each simple factor. Eq. (15) cannot be used for algebras with a degenerate Cartan metric, which are irrelevant to quantum field theory where eq. (15) applies, but are of central interest in gauged supergravities.[7]

Condition (10) can also be read as a condition of the admissible gaugings for a given metric. This is more commonly the case in the context of extended supergravity theories where the abelian (ungauged) theory has a kinetic metric dictated by the electric-magnetic duality group G of the theory. Eq. (10) should then be regarded as the invariance of the metric $\mathcal{N}_{AB}$ under gauge transformations which belong to a subalgebra of the global symmetry group G of the kinetic terms (9). The structure constants $f_{AB}{}^{C}$ are tensors with mixed symmetries under this symmetry group.

3 Vector and scalar fields in supergravity theories

All N–extended four-dimensional supergravity theories ($4N$ supercharges) include vector fields either in the gravity supermultiplet ($2 \leq N \leq 8$) or in the vector supermultiplet ($1 \leq N \leq 4$)[8]. The Lagrangian describing the supersymmetric interactions trivially possesses the required abelian gauge symmetry $U(1)^m$ (m is the number of vector fields). It also includes in general non-minimal couplings of the gauge potentials to other fields (interactions depending on their curvature instead of minimal couplings with covariant derivatives). In particular, the kinetic metric $\mathcal{N}_{AB}$ may become a function of scalar fields.

The presence of a number of gauge fields opens the possibility of extending the abelian symmetry to a non-abelian gauge algebra, with the required covariantizations

[7]For a more complete discussion of the consequences of condition (10), see Ref. [9].

[8]Tensor or linear multiplets will not be considered here.

of derivatives, couplings and transformations used in the dynamical description of the fields. The procedure of extending $U(1)^m$ to a non-abelian algebra is the *gauging* of the theory.[9]

Four-dimensional supergravity theories also include scalar fields in their gravity multiplet ($4 \leq N \leq 8$), in their vector multiplet ($2 \leq N \leq 4$), in the $N = 2$ hypermultiplet or in the $N = 1$ chiral multiplet. In each case, the scalar kinetic Lagrangian is characterized by a specific sigma-model structure,

$$-\frac{1}{2} e\, g_{ij}\, (D_\mu \varphi^i)(D^\mu \varphi^j),$$

where the possible choices for the metric $g_{ij}(\varphi^k)$ strongly depend on the supermultiplets under consideration. We will be mostly concerned here with $N = 4$ supergravity coupled to vector supermultiplets, or with the $N = 8$ theory. In these cases, the scalar fields live on a coset G/H. In addition, G is a duality symmetry of the theory (see below): it acts as a global symmetry of the field equations and Bianchi identities of the abelian gauge fields. Its maximal compact subgroup H is a (local) linear symmetry of the scalar interactions. In the $N = 4$ theory with n vector multiplets, the scalar manifold is the coset

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(n,6)}{SO(n) \times SO(6)},$$

with dimension $2+6n$. The theory includes $6+n$ vector fields and the duality symmetry $SU(1,1) \times SO(6,n) \subset Sp(12 + 2n, R)$ acts (linearly) on the $12 + 2n$ gauge curvatures and on their duals. Gauging the theory does in general break the duality symmetry.

Notice for future use that the truncation to $N = 4$ of the $N = 8$ (ungauged) theory, with duality symmetry $E_{7,7}$ leads to six vector multiplets, with embedding

$$E_{7,7} \supset SU(1,1) \times SO(6,6)$$

$$\mathbf{56} = (\mathbf{2}, \mathbf{12}) + (\mathbf{1}, \mathbf{32'}), \qquad \mathbf{133} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{66}) + (\mathbf{2}, \mathbf{32}),$$

where $\mathbf{32}$ and $\mathbf{32'}$ are the two spinors of $SO(6,6)$.

4 Electric-magnetic duality

In the ungauged version of $N \geq 4$ supergravity (in four dimensions), the symmetry group G of the scalar manifold G/H is also a duality symmetry acting on the abelian gauge curvatures and their duals. This is possible because the couplings of gauge fields are of non-minimal type depending on the curvatures only.[10] A nonabelian gauging

[9] Even if allocating non-trivial abelian charges to some fields of the theory, with the appropriate covariantization of derivatives, is also a gauging.

[10] The supergravity theories discussed here describe interactions with at most two derivatives. They include terms linear and quadratic in $F^A_{\mu\nu}$ and $\tilde{F}^A_{\mu\nu}$.

of the theory breaks the duality symmetry: the explicit dependence on A_μ^A of gauge curvatures and covariant derivatives of (non-singlet) fields modifies the field equations and Bianchi identities.

If the Lagrangian depends on the vector fields A_μ^A ($A = 1, \ldots, m$) only through the abelian curvature $F_{\mu\nu}^A$, the Euler-Lagrange equation

$$\partial^\mu \tilde{G}_{\mu\nu}^A = 0, \qquad \tilde{G}_{\mu\nu}^A \equiv -2\frac{\delta\mathcal{L}}{\delta F^{A\,\mu\nu}} \tag{16}$$

and the Bianchi identity

$$\partial^\mu \tilde{F}_{\mu\nu}^A = 0, \qquad \tilde{F}_{\mu\nu}^A = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{A\,\rho\sigma} \tag{17}$$

are left invariant by $Gl(2m, R)$ linear transformations

$$(F_{\mu\nu}^A, G_{\mu\nu}^A) \qquad \longrightarrow \qquad (F_{\mu\nu}^{A\,\prime}, G_{\mu\nu}^{A\,\prime}).$$

In addition, the Lagrangian $\mathcal{L}$ should simultanously transform into $\mathcal{L}'$ in such a way that

$$\tilde{G}_{\mu\nu}^{A\,\prime} = -2\frac{\delta\mathcal{L}'}{\delta F^{A\,\mu\nu\,\prime}}.$$

A similar condition of covariance applies to other field equations. As a consequence, Gaillard and Zumino [10] showed that the duality group reduces to $Sp(2m, R)$ and that the Lagrangian has a universal (although implicit) expression,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^A\tilde{G}^{A\,\mu\nu} + \mathcal{L}_{inv.}, \tag{18}$$

where $\mathcal{L}_{inv.}$ depends in general of all fields. The true duality group $G \subset Sp(2m, R)$ of the theory leaves then $\mathcal{L}_{inv.}$ invariant. It is in general smaller that $Sp(2m, R)$ since G should also act on other fields to lead to a nontrivial invariant $\mathcal{L}_{inv.}$.

In N–extended supergravity theories with scalars on coset G/H, G is expected to be the electric-magnetic duality group of the ungauged theory. Hence, $G = E_{7,7}$ for the $N = 8$ theory, which has 28 vector fields. The embedding $Sp(56, R) \supset G$ is simply $\mathbf{56} = \mathbf{56}$. For the $N = 4$ theory with n vector multiplets (and $m = n + 6$ vector fields), $G = SU(1,1) \times SO(n,6)$. The embedding $Sp(12 + 2n, R) \supset G$ is then defined by $\mathbf{12} + \mathbf{2n} = (\mathbf{2}, \mathbf{6} + \mathbf{n})$.

Transformations of the duality group relate *different* Lagrangians with however equivalent field equations. In particular, the role of some (*electric*) gauge fields $F_{\mu\nu}$ may be exchanged with their (*magnetic*) duals $\tilde{F}_{\mu\nu}$. Equivalent dynamical field equations are then described by Lagrangians corresponding to various choices of *symplectic frames* and related by duality transformations, which are symplectic $Sp(2m, R)$ transformations.

Starting with an ungauged supergravity with m gauge fields, one can in principle choose to gauge a non-abelian algebra acting on the curvatures $F_{\mu\nu}$ *and* on the duals, provided certain consistency conditions are applied. In particular, the resulting theory should propagate the same degrees of freedom required by supersymmetry as the ungauged theory. This approach recently developed by de Wit, Samtleben and Trigiante [11] allows to discuss gaugings of extended supergravity theories in very general terms. In addition, this formulation respects the electric-magnetic duality symmetry which is so useful in the construction of extended supergravity Lagrangians. One should however notice that gauged supergravities described in the literature are necessarily written in a certain symplectic frame (which breaks the electric-magnetic duality symmetry). A comparison with the general gauging procedure derived with the method of ref. [11] requires to find the duality transformation relating both formulations.

In Section 5, we will briefly describe this general method and its use in the context of gauged $N = 4$ supergravity.

4.1 The duality algebra $Sp(2m, R)$

The Lie algebra of $Sp(2m, R)$, in the fundamental representation $\boldsymbol{2m}$, can be represented by real matrices P such that

$$P^\tau \Omega + \Omega P = 0, \tag{19}$$

where $\Omega = -\Omega^\tau = -\Omega^{-1}$ is the symplectic metric. The standard choice is

$$\Omega = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{20}$$

(I_m is the identity matrix in m dimensions), but this is not necessarily the most convenient for our purposes. Clearly, ΩP is symmetric, or, with choice (20),

$$P = \begin{pmatrix} A & B \\ C & -A^\tau \end{pmatrix} \tag{21}$$

with A arbitrary, B and C symmetric.

Some subalgebras will be useful. Firstly, $Sp(2m, R) \supset Sp(2, R)^m$. The $3m$ parameters of this subalgebra correspond to diagonal matrices A, B and C. Secondly, $Sp(2m, R) \supset SU(1, 1) \times SO(p, q)$ for all p and q such that $p + q = m$. This subalgebra is directly relevant to $N = 4$ supergravity and will be described in the next paragraph. Thirdly, the maximal compact subalgebra $U(m)$ corresponds to

$$P_{U(m)} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = -P_{U(m)}{}^\tau, \tag{22}$$

with B symmetric and A antisymmetric.

As a duality algebra, $Sp(2m,R)$ acts on $F^A_{\mu\nu}$ and $G^A_{\mu\nu}$ according to

$$\delta(\,F\,,\,G\,) = (\,AF + BG\,,\,-A^\tau G + CF).$$

An *electric duality* is such that starting with a Lagrangian depending on the $F^A_{\mu\nu}$ only (and not on the $\tilde{F}^A_{\mu\nu}$), the transformed theory also does not depend on the (transformed) $G^A_{\mu\nu}$. Since the gauge algebra is included in $Sp(2m,R)$, an electric gauging of a given Lagrangian uses only generators with $B=0$. Notice that C does not need to be zero. As a simple example, consider $m=1$ and theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F_{\mu\nu}\tilde{G}^{\mu\nu}, \qquad\qquad (\,\tilde{G}^{\mu\nu} = F^{\mu\nu}\,). \tag{23}$$

The variation under $Sp(2,R)$ with $B=0$ is

$$\delta\mathcal{L} = -\frac{1}{4}\,C\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{24}$$

i.e. C generates *axionic symmetries*, which can in principle be gauged. Standard gaugings of quantum field theory, as outlined in Section 2, are invariances of the Lagrangian with $B=C=0$. In this case, the gauged algebra is included in the $SO(m)$ global symmetry of gauge kinetic terms[11] generated by the antisymmetric A.

4.2 The duality algebra of $N=4$ supergravity: $SU(1,1)\times SO(n,6)$

For a theory with m vector fields, the electric-magnetic duality algebra G is a subalgebra of $Sp(2m,R)$ and the gauge algebra is included in G. As already mentioned, in the case of $N=4$ supergravity with n vector supermultiplets (*i.e.* with $m=6+n$ vector fields),

$$G = SU(1,1)\times SO(n,6) \subset Sp(12+2n,R).$$

To realize the $SU(1,1)\times SO(p,q)$ subalgebra of $Sp(2m,R)$, it is useful to replace the symplectic metric (20) by

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \eta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \tag{25}$$

where $\eta = \eta^\tau = \eta^{-1}$ is the $SO(p,q)$ metric with p eigenvalues $+1$ and $q=m-p$ eigenvalues -1. Its diagonal form would be $\eta = \mathrm{diag}(I_p,-I_q)$ but we will not need to

[11] The symmetry is $SO(m)$ and not $SO(p,q)$ because positivity of the kinetic metric is required in quantum field theory.

assume that η is diagonal. An element of the $SO(p,q)$ algebra (in the vector representation $\boldsymbol{m}$) is a matrix O such that ηO is antisymmetric. In other words, $O = \eta M$ with M antisymmetric. The solution of the $Sp(2m,R)$ defining equation (19) is now

$$P = \begin{pmatrix} \eta A & \eta B \\ \eta C & -\eta A^\tau \end{pmatrix}, \tag{26}$$

again with A arbitrary, B and C symmetric. Choosing $A = \alpha\eta$, $B = \beta\eta$ and $C = \gamma\eta$ (α, β and γ are real numbers) leads to the $Sp(2,R) \sim Sl(2,R) \sim SU(1,1)$ subalgebra generated by

$$P_{SU(1,1)} = \begin{pmatrix} \alpha I_m & \beta I_m \\ \gamma I_m & -\alpha I_m \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \otimes I_m. \tag{27}$$

It commutes with elements of $Sp(2m,R)$ of the form

$$P_{SO(p,q))} = \begin{pmatrix} \eta M & 0 \\ 0 & \eta M \end{pmatrix} = I_2 \otimes \eta M, \qquad\qquad M^\tau = -M, \tag{28}$$

which generate the $SO(p,q)$ algebra. Strictly speaking, $SO(p,q)$, with its block-diagonal form, is not an electric-magnetic duality symmetry. In $N=4$ supergravity, it is a global symmetry of the (ungauged) action.

The $SU(1,1)\times SO(p,q)$ decomposition of $Sp(2m,R)$ is then as follows. In expression (26), split the matrices according to

$$A = A_- + A_0 + \frac{1}{m}\operatorname{Tr}(\eta A)\,\eta, \qquad B = B_0 + \frac{1}{m}\operatorname{Tr}(\eta B)\,\eta, \qquad C = C_0 + \frac{1}{m}\operatorname{Tr}(\eta C)\,\eta,$$

with A_- antisymmetric while the symmetric matrices A_0, B_0 and C_0 have zero "η–trace", $0 = \operatorname{Tr}(\eta A_0) = \operatorname{Tr}(\eta B_0) = Tr(\eta C_0)$. This leads to

$$P = \frac{1}{m}\begin{pmatrix} \operatorname{Tr}(\eta A) & \operatorname{Tr}(\eta B) \\ \operatorname{Tr}(\eta C) & -\operatorname{Tr}(\eta A) \end{pmatrix} \otimes I_m \;+\; I_2 \otimes \eta A_- \;+\; \begin{pmatrix} \eta A_0 & \eta B_0 \\ \eta C_0 & -\eta A_0 \end{pmatrix}. \tag{29}$$

The first two terms generate the $H \equiv SU(1,1) \times SO(p,q)$ subalgebra. The third one includes the generators of the coset $Sp(2m,R)/G$ (with $3[\frac{1}{2}m(m+1)-1]$ parameters), which is absent in the duality symmetry of $N=4$ supergravity, and then also in the gauge algebra of $N=4$ supergravity. The compact $U(1)$ subgroup of $SU(1,1)$ is

$$P_{U(1)} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \otimes I_m\,. \tag{30}$$

This $U(1)$ defines the complex basis required by the Kähler structure of the $N=4$ supergravity dilaton. If on the real basis (F,G),

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = P \begin{pmatrix} F \\ G \end{pmatrix}$$

as in eq. (29), then $SU(1,1)\times SO(p,q)$ acts on $F\pm iG$ according to

$$\delta\begin{pmatrix} F+iG \\ F-iG \end{pmatrix} = \left\{\begin{pmatrix} ia & b+ic \\ b-ic & -ia \end{pmatrix}\otimes I_m + I_2\otimes\eta A_-\right\}\delta\begin{pmatrix} F+iG \\ F-iG \end{pmatrix} \tag{31}$$

with a,b,c real and the compact $U(1)$ is diagonal.

It may be useful to write component expressions for the generators of $SU(1,1)\times SO(p,q)$ in the basis defined by expression (29). Write then

$$\Big(P_G\Big)_{\alpha I}{}^{\beta J} = \frac{1}{2}\alpha^{\gamma\delta}(T_{\gamma\delta})_\alpha{}^\beta\,\delta_I^J + \frac{1}{2}\alpha^{KL}(T_{KL})_I{}^J\,\delta_\alpha^\beta \tag{32}$$

with $SU(1,1)$ indices $\alpha,\beta,\ldots=1,2$ and $SO(p,q)$ indices $I,J,\ldots=1,\ldots,m$, and real parameters $\alpha^{\gamma\delta}=\alpha^{\delta\gamma}$ and $\alpha^{KL}=-\alpha^{LK}$. Use then

$$\begin{array}{rcll} (T_{\gamma\delta})_\alpha{}^\beta &=& \epsilon_{\gamma\alpha}\delta_\delta^\beta+\epsilon_{\delta\alpha}\delta_\gamma^\beta, & \qquad (\epsilon_{12}=1, \quad \epsilon_{\delta\beta}=-\epsilon_{\beta\delta}), \\ (T_{KL})_I{}^J &=& \eta_{KI}\delta_L^J-\eta_{LI}\delta_K^J & \end{array} \tag{33}$$

as generators of $SU(1,1)$ and $SO(p,q)$. Notice that $SU(1,1)$ is represented in a real space. In this basis, the symplectic metric (25) reads

$$(\Omega)_{\alpha I\,\beta J} = \epsilon_{\alpha\beta}\,\eta_{IJ}. \tag{34}$$

As it should, it is left invariant by generators (33).

With choice (33) of $SU(1,1)\times SO(n,6)$ generators, an electric gauging does not involve the $SU(1,1)$ generator $T_{\alpha=2\,\beta=2}$ and axionic symmetries have generators involving $T_{\alpha=1\,\beta=1}$.

A gauging of $N=4$ supergravity proceeds then by selecting linear combinations of the generators (33) to represent the embedding of the gauge algebra inside the duality algebra G.

5 Gauging: the embedding tensor

Very schematically, the gauging procedure developed in ref. [11] is as follows. Assume that the duality algebra $G\subset Sp(2m,R)$ of a given theory has generators T_A acting in the fundamental representation of $Sp(2m,R)$ (indices $M,N,P,\ldots$), with

$$[T_A,T_B]=f_{AB}{}^C\,T_C, \qquad f_{AB}{}^C=-f_{BA}{}^C, \qquad (T_A)_{]M}{}^P\Omega_{N]P}=0. \tag{35}$$

Since the gauge algebra is a subalgebra of G, we may choose combinations

$$X_M = \Theta_M{}^A\,T_A\,, \tag{36}$$

where $\Theta_M{}^A$ is the *embedding tensor*, to define its generators. The index M reflects the embedding [adjoint of the gauge algebra] $\subset$ [fundamental of $Sp(2m, R)$]. Closure of the gauge algebra is the equation

$$[X_M, X_N] = X_{MN}{}^P X_P = X_{MN}{}^P \Theta_P{}^A T_A \,. \tag{37}$$

The constant numbers $X_{MN}{}^P$ form a tensor under G. As usual [see eq. (6)], the matrix elements of the gauge generators are $(X_M)_P{}^N = X_{MP}{}^N$. Eq. (37) implies firstly a linear constraint on the embedding tensor:

$$X_{(MN)}{}^P \Theta_P{}^A = 0. \tag{38}$$

The antisymmetric part $X_{[MN]}{}^P \Theta_P{}^A$ defines the structure constants of the gauge algebra, $X_{[MN]}{}^P X_P = f_{MN}{}^P X_P$. The closure of the gauge algebra (37) also implies quadratic constraints on the embedding tensor which can be regarded as generalized Jacobi identities.

Consistency of the procedure, including the existence of a Lagrangian propagating the correct number of states (*i.e.* m states with helicities ± 1), actually implies somewhat stronger conditions [11]. Specifically, the invariance of the symplectic metric $X_{M[N}{}^Q \Omega_{P]Q} = 0$ is supplemented by

$$X_{(MN}{}^Q \Omega_{P)Q} = 0. \tag{39}$$

Hence,

$$\hat{X}_{MNP} \equiv X_{MN}{}^Q \Omega_{PQ} \tag{40}$$

is a G–tensor with mixed symmetry: it is symmetric in NP but it is not fully symmetric. From the point of view of $Sp(2m, R)$, the symmetric indices NP are in the adjoint representation and $\hat{X}_{MNP}$ is in the product (adjoint) $\times$ (fundamental), with the fully symmetric tensor projected out.

These consistency conditions were found on the basis of a study of general gaugings of maximal supergravities [12, 13, 14, 15, 16, 17, 18]. In particular, condition (40) is required by supersymmetry of the action.

The application of this general procedure to the gauging of $N = 4$ supergravity coupled to n vector multiplets has been recently described by Schön and Weidner [19]. The gauge generators are linear combinations of the generators of the $N = 4$ duality algebra $G = SU(1,1) \times SO(n, 6)$, as defined in expressions (33):

$$X_{\alpha I} = \frac{1}{2}\, \Theta_{\alpha I}{}^{\beta\gamma}\, T_{\alpha\beta} + \frac{1}{2}\, \Theta_{\alpha I}{}^{JK}\, T_{JK}. \tag{41}$$

In components,

$$X_{\alpha I\, \beta J}{}^{\gamma K} = \Theta_{\alpha I}{}^{\rho\gamma} \epsilon_{\rho\beta}\, \delta_J^K + \Theta_{\alpha I}{}^{LK} \eta_{LJ}\, \delta_\beta^\gamma \,. \tag{42}$$

Alternatively, as in eq. (40),

$$\begin{aligned}\hat{X}_{\alpha I\,\beta J\,\sigma M} &\equiv X_{\alpha I\,\beta J}{}^{\gamma K}\Omega_{\gamma K\,\sigma M} \\ &= \Theta_{I\alpha(\beta\sigma)}\,\eta_{JM} + \Theta_{\alpha I[JM]}\,\epsilon_{\beta\sigma} \;=\; \hat{X}_{\alpha I\,\sigma M\,\beta J}\,.\end{aligned} \tag{43}$$

The last equality follows from the invariance of the symplectic metric. The embedding tensors

$$\Theta_{I\alpha(\beta\sigma)} = \Theta_{\alpha I}{}^{\rho\gamma}\epsilon_{\rho\beta}\epsilon_{\gamma\sigma}, \qquad\qquad \Theta_{\alpha I[JM]} = \Theta_{\alpha I}{}^{LK}\eta_{LJ}\eta_{KM} \tag{44}$$

respectively transform as $(\mathbf{2}\times\mathbf{3},\mathbf{6}+\mathbf{n})$ and $(\mathbf{2},(\mathbf{6}+\mathbf{n})\times\mathbf{Adj})$ of $SU(1,1)\times SO(n,6)$.[12]

In this symplectic frame, an electric gauging corresponds to the condition

$$\Theta_{\alpha I}{}^{\beta=2\,\gamma=2} = 0$$

or to $\Theta_{I\alpha\,(\beta=1\,\gamma=1)} = 0$. It is left invariant by $SO(n,6)$, but not by $SU(1,1)$.

The analysis of $N=8$ supergravity gaugings given in ref. [17] can be used to obtain information of the $N=4$ embedding tensor. The duality group of $N=8$ supergravity is $E_{7,7}$, and the embedding tensor is in the product (fundamental)×(adjoint):

$$\mathbf{56}\times\mathbf{133} = \mathbf{56}+\mathbf{912}+\mathbf{6480}.$$

Consistency of the gauging imposes however that the embedding tensor is in representation **912** only [17]. One of the consistency conditions applied to the embedding tensor is then simply its projection into this representation only. Truncation from $N=8$ to $N=4$ can be performed by removing all representations with $SO(6,6)$ spinor weights in the embedding $E_{7,7}\supset SU(1,1)\times SO(6,6)$:

$$\begin{aligned}\mathbf{56} &\rightarrow (\mathbf{2},\mathbf{12}), & \mathbf{133} &\rightarrow (\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{66}),\\ \mathbf{912} &\rightarrow (\mathbf{2},\mathbf{12})+(\mathbf{2},\mathbf{220}), & \mathbf{6480} &\rightarrow (\mathbf{2},\mathbf{12})+(\mathbf{4},\mathbf{12})+(\mathbf{2},\mathbf{220})+(\mathbf{2},\mathbf{560}).\end{aligned}$$

We then infer that the embedding tensor includes two components transforming as $(\mathbf{2},\mathbf{12})$ and $(\mathbf{2},\mathbf{220})$ only (**220** is the three-index antisymmetric tensor). It will then be expressed as a function of constant tensors $\xi_{\alpha I}$ [representation $(\mathbf{2},\mathbf{12})$] and $f_{\alpha[IJK]}$ [representation $(\mathbf{2},\mathbf{220})$]. Generalization to an arbitrary number of vector multiplets is then simply obtained by replacing $SO(6,6)$ by $SO(n,6)$ and considering the same tensor representations for this algebra.

Explicitly, the projection into the two relevant directions is as follows. First define the three-index antisymmetric tensor[13]

$$f_{\alpha[IJK]} = \frac{1}{3}\Big(\Theta_{\alpha I[JK]}+\Theta_{\alpha J[KI]}+\Theta_{\alpha K[IJ]}\Big). \tag{45}$$

[12] **Adj** refers to the (antisymmetric) adjoint representation of $SO(n,6)$.

[13] $SU(1,1)$ and $SO(n,6)$ indices are moved using $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, η_{IJ} and η^{IJ}. The conventions are $\xi^\alpha_I = \epsilon^{\alpha\beta}\xi_{I\beta}$, $\xi_{I\beta} = \epsilon_{\beta\alpha}\xi^\alpha_I$ and $\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma$.

Write then

$$\Theta_{\alpha I[JK]} = f_{\alpha[IJK]} - \frac{1}{(6+n)-1}\Big(\eta_{IJ}\Theta_\alpha{}^M{}_{[MK]} - \eta_{IK}\Theta_\alpha{}^M{}_{[MJ]}\Big). \tag{46}$$

This linear condition eliminates the unwanted direction which would correspond to representation $(\mathbf{2}, \mathbf{560})$ in the $n = 6$ case relevant to $N = 8$ supergravity. Similarly,

$$\Theta_{I\alpha(\beta\gamma)} = -\frac{1}{3}\Big(\epsilon_{\alpha\beta}\Theta^\sigma{}_{I(\sigma\gamma)} + \epsilon_{\alpha\gamma}\Theta^\sigma{}_{I(\sigma\beta)}\Big) \tag{47}$$

eliminates the unwanted direction $(\mathbf{4}, \mathbf{6}+\mathbf{n})$. This apparently leaves two independent representations $(\mathbf{2}, \mathbf{6}+\mathbf{n})$, with tensors

$$\xi_{\alpha I} \equiv \frac{1}{(6+n)-1}\,\Theta_\alpha{}^M{}_{[MI]} \qquad\qquad \text{and} \qquad\qquad \zeta_{\alpha I} \equiv \frac{1}{3}\,\Theta^\sigma{}_{I(\sigma\alpha)}, \tag{48}$$

while $N = 8$ supergravity predicts a single representation inside the **912** of $E_{7,7}$. Imposing the linear condition (39) leads then to $\xi_{\alpha I} = \zeta_{\alpha I}$ or

$$\hat{X}_{\alpha I\,\beta J\,\sigma M} = f_{\alpha[IJM]}\,\epsilon_{\beta\sigma} - \eta_{JM}(\epsilon_{\alpha\beta}\zeta_{\sigma I} + \epsilon_{\alpha\sigma}\zeta_{\beta I}) - \epsilon_{\beta\sigma}(\eta_{IJ}\zeta_{\alpha M} - \eta_{IM}\zeta_{\alpha J})\,. \tag{49}$$

The embedding tensors $f_{\alpha[IJK]}$ and $\zeta_{\alpha I}$ are submitted to a complicated set of quadratic constraints ensuring closure of the gauge algebra, eq. (37).[14]

Notice that the gauge generators are

$$X_{\alpha I} = \frac{1}{2} f_{\alpha\, I}{}^{JK} T_{JK} - \zeta^J_\alpha T_{IJ} - \zeta^\beta_I T_{\alpha\beta}. \tag{50}$$

Hence, $SU(1,1)$ generators $T_{\alpha\beta}$ are necessarily combined with $SO(n,6)$ generators, the tensor $\zeta_{\alpha I}$ acting as a common parameter.

In the particular case $\zeta_{\alpha I} = 0$ the quadratic conditions reduce to

$$\begin{aligned} 0 &= f_{\alpha R[MN} f_{\alpha PQ]S}\,\eta^{RS}\,, \qquad\qquad \alpha = 1,2 \qquad (\text{no sum on } \alpha) \\ 0 &= f_{1\,MNR} f_{2\,PSQ}\,\eta^{RS} - f_{1\,MNR} f_{2\,PSQ}\,\eta^{RS}\,. \end{aligned} \tag{51}$$

The first line indicates that $f_{1\,IJ}{}^K$ and $f_{2\,IJ}{}^K$ verify separately the Jacobi identity. The second equation indicates that the generators of these two Lie algebras commute. A way to solve these conditions [19] is to start with a semi-simple algebra $\prod \mathcal{G}_{(i)}$, with structure constants $f^{(i)}_{IJ}{}^K$ verifying Jacobi identities for each factor: $f^{(i)}_{[IJ}{}^K f^{(i)}_{L]K}{}^M = 0$. The Cartan metric is such that this algebra can be embedded in $SO(n,6)$. The resulting $f^{(i)}_{IJK} = f^{(i)}_{IJ}{}^L \eta_{LK}$ are then antisymmetric. The quadratic conditions (51) are solved if $f^{(i)}_{1\,IJK}$ and $f^{(i)}_{2\,IJK}$ are proportional for each factor separately. In other words,

$$f_{\alpha\,IJK} = \sum_i c^{(i)}_\alpha f^{(i)}_{IJK} \tag{52}$$

[14] See eqs. (2.20) of ref. [19].

with real vectors $c_\alpha^{(i)} = \lambda^{(i)}(\cos\delta_i, \sin\delta_i)$. The angles α_i correspond to the phases found by de Roo and Wagemans [20, 21] in their analysis of gauged $N = 4$ supergravity.[15]

Hence, in this class of gaugings, the theory depends on antisymmetric *gauging structure constants* f_{RST} related to the structure constants of the gauge algebra by $f_{RST} = f_{RS}{}^U \eta_{UT}$ and submitted to Jacobi identities. With each direction R, a *duality angle* (or *duality phase*) δ_R can be introduced provided the quadratic conditions (51) are verified.

The assumption $\zeta_{\alpha I} = 0$ also implies $\Theta_{I\alpha(\beta\sigma)} = 0$ *i.e.* the gauge algebra does not involve the $SU(1,1)$ generators. Since $SO(n,6)$ is an invariance of the Lagrangian (while $SU(1,1)$ is the true duality symmetry mixing gauge curvatures and duals), this case should be considered as an electric gauging without any shift symmetry included in the gauge algebra.

More general gaugings with in particular $\zeta_{\alpha I} \neq 0$ have not been extensively studied yet.

6 $N = 4$ supergravity

This section summarizes the aspects of the $N = 4$ supergravity theory [22, 23, 24] and of its reduction to $N = 1$ which are useful in discussing $N = 1$ string moduli superpotentials. It mostly concentrates on the bosonic sector and on truncating the theory in a $N = 1$ supergravity formulation, *i.e.* in a Kähler basis for the scalar fields.

The $N = 4$ super-Yang-Mills–supergravity system is most conveniently obtained [25] from the action of the (locally) superconformal $N = 4$ Yang-Mills theory. This approach reveals at the linear level the sigma-model structure of the scalar kinetic terms and the nature of the electric-magnetic duality algebra.

6.1 $N = 4$ conformal supergravity

The starting point of the construction is the action of (locally) superconformal $N = 4$ super-Yang-Mills theory. This theory is obtained by superconformal calculus [26] of two kinds of superconformal multiplets, the (Weyl) multiplet of gauge fields [27] and vector supermultiplets. The superconformal $N = 4$ superalgebra is $SU(2,2|4)$, with bosonic subalgebra

$$SU(2,2) \times SU(4) \simeq SO(4,2) \times SO(6).$$

[15] After rescaling $\lambda^{(i)}$ to one.

The four-dimensional conformal algebra is $SO(4,2)$ while $SU(4) \simeq SO(6)$ is the R–symmetry acting on the four supersymmetries. Notice that there is no additional $U(1)$, as would have been the case for $N \neq 4$, with bosonic subalgebra $SU(2,2) \times SU(N) \times U(1)$. The superalgebra has sixteen supersymmetries and sixteen special supersymmetries. The field theory includes then the following gauge fields:

Field	*Name*	*Symmetry*	*Status*
ω_μ^{ab}	Spin connection	Lorentz	Eliminated
e_μ^a	Vierbein	Translations	Propagating
f_μ^a	Special vierbein	Conformal boosts	Eliminated
b_μ	Dilatation gauge field	Dilatation	Gauge-fixed
$V_\mu{}^i{}_j$	$SU(4)$ gauge fields	$SU(4)$ R-symmetry	Auxiliary
ψ_μ^i	Gravitino	$N=4$ supersymmetry	Propagating
ϕ_μ^i	Special gravitino	$N=4$ special supersymmetry	Eliminated

They are submitted to "curvature constraints" and, in addition, to the Poincaré gauge-fixing conditions of the unnecessary symmetries: conformal boosts, dilatations, and special supersymmetry. The first four lines in the table include the fifteen gauge fields of the conformal algebra. The next three lines include the fifteen $SU(4)$ gauge fields and the eight supersymmetry and special supersymmetry gauge fields. Fermions are Weyl vector-spinors in the complex representation **4** of $SU(4)$ (or in its conjugate):

$$\gamma_5 \psi_\mu^i = \psi_\mu^i \equiv (\psi_{\mu\, i})^*, \qquad\qquad \gamma_5 \phi_\mu^i = -\phi_\mu^i \equiv -(\phi_{\mu\, i})^*.$$

The status column refers to the Poincaré theory, after imposing the curvature constraints (which eliminate ω_μ^{ab}, f_μ^a and ϕ_μ^i) and after the Poincaré gauge-fixing conditions (which apply in particular to b_μ). Off-shell and on-shell (gauge) degrees of freedom are then as follows:

Field	*Off-shell*	*On-shell*
e_μ^a	$4\times4-4-6-1=5$	2
b_μ	$4-4=0$	0
$V_\mu{}^i{}_j$	$15\times(4-1)=45$	0
ψ_μ^i	$4\times(4^2-4-4)=32$	8
Total:	50_B+32_F	2_B+8_F

The subtractions in the off-shell counting refer to general coordinate transformations (GCT), local Lorentz and dilatation symmetries for the vierbein, conformal boosts for b_μ, supersymmetries and special supersymmetries for the gravitinos. The on-shell counting is for information only since it refers to a specific action.

To complete the Weyl $N=4$ supermultiplet ($128_B + 128_F$ off-shell fields), we need the following $78_B + 96_F$ matter (*i.e.* non-gauge) fields:

- The $SU(1,1)/U(1)$ dilaton sector: two propagating bosons (see paragraph 6.4).
- Complex auxiliary scalars $E_{ij} = E_{ji}$ in representation **10** of $SU(4)$, *i.e.* twenty bosons.
- Thirty-six bosons in fields:

$$T_{\mu\nu}{}^{ij} = -T_{\nu\mu}{}^{ij} = -T_{\mu\nu}{}^{ji}, \qquad\qquad T_{\mu\nu}{}^{ij} = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}T_{\rho\sigma}{}^{ij}.$$

  These fields are then in representation $(\mathbf{1},\mathbf{3})$ of the Lorentz group. They are complex and in **6** of $SU(4)$.
- Twenty scalar bosons in (real) representation **20** of $SU(4)$:

$$D^{ij}{}_{kl} = \frac{1}{4}\epsilon^{ijmn}\epsilon_{klpq}D^{pq}{}_{mn}, \qquad (D^{ij}{}_{kl})^* = D^{kl}{}_{ij} \equiv D_{ij}{}^{kl}, \qquad D^{ij}{}_{kj} = 0.$$

- Four Weyl spinors $\Lambda_i = \gamma_5\Lambda_i$ in representation $\overline{\mathbf{4}}$ of $SU(4)$ (sixteen fermions).
- Twenty Weyl spinors $\chi_{ij}{}^k = -\gamma_5\chi_{ij}{}^k$ in (complex) representation $\overline{\mathbf{20'}}$ of $SU(4)$ (eighty fermions):

$$\chi_{ij}{}^k = -\chi_{ji}{}^k, \qquad \chi_{ij}{}^j = 0.$$

Out of the 128_B+128_F (off-shell) fields of the Weyl multiplet, the vierbein, the complex dilaton, the four gravitino and the four spinors Λ_i propagate in the Poincaré theory. On-shell, these fields describe $4_B + 16_F$ of the $16_B + 16_F$ states included in Poincaré pure $N = 4$ supergravity. The missing twelve bosons are provided by six gauge fields from $N = 4$ vector multiplets used to impose the gauge-fixing conditions leading to the Poincaré theory.

6.2 Vector multiplets

The Weyl multiplet of $N = 4$ superconformal gauge fields exists in an off-shell formulation. On the contrary, the vector multiplet only exists on-shell and the coupled supergravity system is then a partially off-shell, partially on-shell construction. As usual, since super-Yang-Mills theory is (globally) conformal invariant, the vector multiplets of superconformal and Poincaré supergravities are identical: $A^R_\mu, \psi^R_i, \phi^R_{ij}$. The gauginos are Weyl spinors in representation **4**, $\psi^R_i = \gamma_5\psi^R_i = (\psi^{iR})^*$. The scalars are complex and in (self-conjugate) representation **6**:

$$\phi^R_{ij} = -\phi^R_{ji} = -\frac{1}{2}\epsilon_{ijkl}\phi^{R\,kl}, \qquad\qquad \phi^{R\,ij} \equiv (\phi^R_{ij})^*. \tag{53}$$

Index R labels the vector multiplets. Each vector multiplet has $8_B + 8_F$ on-shell states. The globally $N = 4$ supersymmetric Lagrangian for the vector multiplets is simply

$$\mathcal{L} = \eta_{RS} \left[-\frac{1}{4} F^R_{\mu\nu} F^{\mu\nu\, S} + \frac{1}{2} \overline{\psi}^{R\, i} \gamma^\mu \partial_\mu \psi^S - \frac{1}{4} (\partial^\mu \phi^{R\, ij})(\partial_\mu \phi^S_{ij}) \right], \tag{54}$$

where η_{RS} is the constant kinetic metric.

Pure $N = 4$ Poincaré supergravity is then obtained by gauge-fixing of the superconformal theory with six vector multiplets. And $N = 4$ Poincaré supergravity coupled to n vector multiplets follows from the superconformal theory with $n + 6$ vector multiplets. The gauge fixing conditions affect the scalar fields ϕ^R_{ij}, which have non-trivial Weyl (dilatation) weights, and it turns out that they can only be solved if η_{RS} is the $SO(n, 6)$ metric,[16] the six negative eigenvalues being associated with the directions of the six compensating multiplets.

In the supergravity Lagrangian, the kinetic metric of gauge fields receives contributions depending on ϕ^R_{ij} from the elimination of the auxiliary tensor fields $T_{\mu\nu}{}^{ij}$ and a direct coupling to the supergravity dilaton dictated by the electric-magnetic duality $SU(1, 1)$. The gauged supergravity theory has a scalar potential which adds a contribution produced by the elimination of the auxiliary scalars E_{ij} to a direct contribution analogous to the D–term potential of $N = 1$ supergravity.

6.3 The scalar constraints

The vector-multiplet scalar fields ϕ^R_{ij} are submitted to two kinds of constraints. Firstly, the auxiliary fields $D^{ij}{}_{kl}$ appear only linearly in the superconformal Lagrangian. Their field equations are then constraints. They are in representation **20** and they couple to a quadratic product of vector-multiplet scalars via the action term

$$\frac{e}{4} D^{ij}{}_{kl} \phi^R_{ij} \phi^{kl}_R, \qquad\qquad \phi^{ij}_R = \eta_{RS} \phi^{ij\, S}.$$

Their field equations imply then that the quadratic (symmetric) product of ϕ^R_{ij} does not have a component in representation **20**. Since $\mathbf{6} \times \mathbf{6} = \mathbf{1}_S + \mathbf{15}_A + \mathbf{20}_S$, only the singlet component remains and the constraint is then

$$\phi^{ij}_R \phi^R_{kl} = \frac{1}{12} \phi^{mn}_R \phi^R_{mn} \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k \right). \tag{55}$$

Secondly, the superconformal Lagrangian includes the Einstein term[17]

$$\frac{1}{12} \left(\phi^{mn}_R \phi^R_{mn} \right) eR.$$

[16] For the vector representation: R, S are indices for representation $\boldsymbol{n+6}$ of $\boldsymbol{SO(n, 6)}$.

[17] The canonically-normalized Einstein Lagrangian is $-\frac{1}{2\kappa^2} eR$.

It will prove convenient to define

$$A = -\frac{1}{6}\,\phi_R^{mn}\phi_{mn}^R, \tag{56}$$

so that fixing dilatation symmetry amounts to require $A = \kappa^{-2}$ (Einstein frame condition). One then finds in the Poincaré theory

$$\phi_R^{ij}\phi_{kl}^R = -\frac{1}{2\kappa^2}\left(\delta_k^i\delta_l^j - \delta_l^i\delta_k^j\right). \tag{57}$$

Altogether, twenty-one scalar fields have been eliminated by these constraints. The local $SU(4)$ symmetry can then be used to eliminate another fifteen scalars which, together with those eliminated by the constraints, form the scalar sector of six vector multiplets.

The scalar fields ϕ_{ij}^R are in representation $(\mathbf{6}, \mathbf{n}+\mathbf{6})$ of $SO(6)\times SO(n,6)$ and the constraints (55) and (57) are invariant under this group [$SO(6)$ is the R–symmetry, which is a local symmetry of scalar kinetic terms]. This structure produces a sigma-model structure $SO(n,6)/SO(6)\times SO(n)$ for vector-multiplet scalars [23].

6.4 The $N = 4$ supergravity dilaton and duality symmetry

This is the $SU(1,1)/U(1)$ sector of the $N = 4$ supergravity multiplet. Introduce two complex scalar fields φ_α, $\alpha = 1, 2$, and the constraint

$$1 = |\varphi_1|^2 - |\varphi_2|^2 \equiv \varphi^\alpha\varphi_\alpha, \tag{58}$$

with $\varphi^1 = \varphi_1^*$, $\varphi^2 = -\varphi_2^*$. This constraint is clearly invariant under $U(1,1) = SU(1,1)\times U(1)$. The abelian factor is the global phase of φ_α.

The standard solution to eq. (58) uses a complex scalar field S with relations[18]:

$$\begin{aligned} &\varphi_1 - \varphi_2 = \sqrt{\frac{2}{S+\overline{S}}}, \qquad && \varphi_1 + \varphi_2 = \sqrt{\frac{2}{S+\overline{S}}}\,S. \\ &S = \frac{\varphi_1+\varphi_2}{\varphi_1-\varphi_2}, && \frac{\varphi_1}{\varphi_2} = \frac{S+1}{S-1}. \end{aligned} \tag{59}$$

A kinetic Lagrangian invariant under global $SU(1,1)$ and local $U(1)$ transformations is

$$\mathcal{L} = \frac{e}{\kappa^2}\,(D_\mu\varphi^\alpha)(D^\mu\varphi_\alpha) \qquad D_\mu\varphi_\alpha = (\partial_\mu + iA_\mu)\varphi_\alpha, \tag{60}$$

A_μ being the $U(1)$ gauge field. Eliminating A_μ with the constraint (58) leads to

$$A_\mu = i\varphi^\alpha\partial_\mu\varphi_\alpha = -i\varphi_\alpha\partial_\mu\varphi^\alpha = -(S+\overline{S})^{-1}\partial_\mu\,\mathrm{Im}\,S \tag{61}$$

[18] There is at this stage an ambiguity $S \leftrightarrow \overline{S}$ in the solution.

and the Lagrangian (60) rewrites as

$$\mathcal{L} = -(S+\overline{S})^{-2}\,(\partial_\mu \overline{S})(\partial^\mu S) = -\frac{\partial^2 K_S}{\partial S \partial \overline{S}}\,(\partial_\mu \overline{S})(\partial^\mu S), \tag{62}$$

where the dilaton Kähler potential is

$$K_s = -\ln(S+\overline{S}). \tag{63}$$

In the conformal supergravity Lagrangian, the prefactor κ^{-2} in the kinetic Lagrangian (60) is replaced by the field-dependent quantity A defined in eq. (56). Expression (60) appears then in the Poincaré theory written in the Einstein frame.

In the complex basis defined by the φ_α, a $SU(1,1)$ transformation is

$$\varphi_\alpha \quad \longrightarrow \quad (U\varphi)_\alpha, \qquad\qquad U = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}, \qquad |A|^2 - |B|^2 = 1. \tag{64}$$

Infinitesimally

$$U = I + u, \qquad\qquad u = \begin{pmatrix} ia & b+ic \\ b-ic & -ia \end{pmatrix}, \qquad (a,b,c \text{ real}). \tag{65}$$

[Compare with the $SU(1,1)$ contribution in eq. (31)]. The matrix

$$\Phi = \begin{pmatrix} \varphi_1 & \varphi_2^* \\ \varphi_2 & \varphi_1^* \end{pmatrix} \tag{66}$$

is unimodular with constraint (58) and transforms under $SU(1,1)$ according to

$$\Phi \quad \longrightarrow \quad U\Phi, \qquad\qquad |A|^2 - |B|^2 = 1. \tag{67}$$

It parameterizes an element of $SU(1,1)$.

The $SU(1,1)$ symmetry of the dilaton sector and the $SU(1,1)$ electric-magnetic duality algebra are independent. They can however be identified to construct the duality-invariant coupling of the supergravity dilaton to gauge kinetic terms. For instance, in the notation used in section 4.2 [eq. (31)], the quantity

$$\Phi^{-1}\begin{pmatrix} F+iG \\ F-iG \end{pmatrix}$$

is an invariant if both $SU(1,1)$ symmetries are identified.

It may be useful to rewrite the $SU(1,1)$ matrix U in terms of three real parameters:

$$\begin{aligned} A &= \tfrac{1}{2}e^{-i\delta}(s+1/s-it) &= \tfrac{1}{2}(a+d)+\tfrac{i}{2}(b-c), \\ B &= \tfrac{1}{2}e^{+i\delta}(-s+1/s+it) &= \tfrac{1}{2}(a-d)-\tfrac{i}{2}(b+c), \end{aligned} \tag{68}$$

with $ad - bc = 1$.

The $SU(1,1)$ transformation of S using the second parametrization in eqs. (68) is

$$S \quad \longrightarrow \quad \frac{aS - ib}{icS + d}, \qquad\qquad ad - bc = 1. \tag{69}$$

It acts on the Kähler potential (63) with a Kähler transformation. In terms of δ, s and t, one has:

$$S \qquad \longrightarrow \qquad \frac{1}{s^2}\, S_\delta - i\frac{t}{s}, \qquad\qquad S_\delta \equiv \frac{\cos\delta\, S - i\sin\delta}{-i\sin\delta\, S + \cos\delta}. \tag{70}$$

In particular:

$$\delta = 0,\ t = 0: \quad S \longrightarrow \frac{1}{s^2}\, S, \qquad\qquad \delta = 0,\ s = 1: \quad S \longrightarrow S - it. \tag{71}$$

Parameters s and t refer respectively to scaling and axionic shift. The angle δ generates $U(1) \subset SU(1,1)$ transformations including S–inversion (for $\delta = \pi/2, 3\pi/2$).

The electric-magnetic duality symmetry of ungauged $N = 4$ supergravity is generated by the $SU(1,1)$ algebra (27) which commutes with the global symmetry $SO(n,6)$ of gauge kinetic terms. Gauging the theory destroys this $SO(n,6)$ symmetry and each gauge curvature and its dual acquire in principle their own $Sp(2,R) \sim SU(1,1)$ duality algebra, using the $Sp(2,R)^n$ subgroup of $Sp(2n,R)$. The $SU(1,1)$–invariant gauge kinetic terms constructed with the dilaton matrix Φ, as outlined above, may then contain up to $3n$ free parameters. These include n scaling parameters s_R which can be absorbed by field redefinitions, n parameters t_R related to shift symmetries and n duality angles δ_R. The presence of these angles does affect the consistency conditions on the gauging since they explicitly appear in the (dilaton-dependent) kinetic metric of the gauge fields.

7 $Z_2 \times Z_2$ reduction to $N = 1$ supergravity

There are many ways to reduce supersymmetry from $N = 4$ to $N = 1$. In these notes, the example of a $Z_2 \times Z_2$ reduction is considered, as in string orbifolds with the same point group. This truncation leads to a moduli sector with seven chiral multiplets S, T_A, U_A ($A = 1,2,3$) for all string compactifications and compatible orientifolds and D–brane systems. We can also include an arbitrary number of matter multiplets, generically denoted by Z_A^I, ($I = 1,\ldots,n_A$). The $N = 4$ sigma-model $[SU(1,1)/U(1)] \times [SO(n,6)/SO(n) \times SO(6)]$ reduces to the Kähler manifold

$$M_{Z_2\times Z_2} = \frac{SU(1,1)}{U(1)} \times \prod_{A=1}^{3} \frac{SO(2,2+n_A)}{SO(2)\times SO(2+n_A)} \tag{72}$$

($\sum_A n_A = n$). Since

$$\frac{SO(2,2)}{SO(2)\times SO(2)} = \frac{SU(1,1)}{U(1)} \times \frac{SU(1,1)}{U(1)}\,, \tag{73}$$

each complex modulus is associated to an $SU(1,1)/U(1)$ structure in the absence of further Z^I_A fields. In the Lagrangian, the truncation is performed by first rewriting the scalar fields in an $SU(3)$ basis,

$$\phi^{RA} \equiv \phi^{RA4}\,, \qquad \phi^R_A = (\phi^{RA})^* = \frac{1}{2}\epsilon_{ABC}\phi^{RBC}\,, \qquad (A,B,\ldots = 1,2,3)\,. \tag{74}$$

In the $N=4$ supergravity multiplet, the three $SU(3)$ non-singlet gravitino and vector $N=1$ multiplets are then truncated. Similarly, the scalar fields ϕ^R_{ij} submitted to constraints (57) are truncated to $N=1$ multiplets according to the $Z_2\times Z_2$ action on the $SU(3)$ and $SO(n,6)$ indices A and R, as in the sigma model truncation (72). We then introduce three sets of $4+n_A$ complex scalars that we denote by

$$\sigma^1_A,\ \sigma^2_A,\ \rho^1_A,\ \rho^2_A,\ \chi^I_A, \qquad A=1,2,3, \qquad I=1,\ldots,n_A\,. \tag{75}$$

They are submitted to the $Z_2\times Z_2$ truncation of the constraints (57), which reads for each $A=1,2,3$[19]

$$\begin{aligned} |\sigma^1_A|^2 + |\sigma^2_A|^2 - |\rho^1_A|^2 - |\rho^2_A|^2 - \textstyle\sum_I |\chi^I_A|^2 &= 1/2\,, \\ (\sigma^1_A)^2 + (\sigma^2_A)^2 - (\rho^1_A)^2 - (\rho^2_A)^2 - \textstyle\sum_I (\chi^I_A)^2 &= 0\,, \end{aligned} \tag{76}$$

Their invariance is $SO(2,2+n_A)$ and they lead to the sigma-model structure (72). These equations are solved in this basis by:

$$\begin{aligned} \sigma^1_A &= \frac{1}{2}\frac{1+T_AU_A-(Z^I_A)^2}{[Y(T_A,U_A,Z^I_A)]^{1/2}}\,, & \sigma^2_A &= \frac{i}{2}\frac{T_A+U_A}{[Y(T_A,U_A,Z^I_A)]^{1/2}}\,, \\ \rho^1_A &= \frac{1}{2}\frac{1-T_AU_A+(Z^I_A)^2}{[Y(T_A,U_A,Z^I_A)]^{1/2}}\,, & \rho^2_A &= \frac{i}{2}\frac{T_A-U_A}{[Y(T_A,U_A,Z^I_A)]^{1/2}}\,, \\ \chi^I_A &= i\,\frac{Z^I_A}{[Y(T_A,U_A,Z^I_A)]^{1/2}}\,. \end{aligned} \tag{77}$$

These expressions depend on the real quantity

$$Y(T,U,Z^I) = (T+\overline{T})(U+\overline{U}) - \sum_I (Z^I + \overline{Z}^I)^2\,. \tag{78}$$

As expected, the constraints eliminate six complex scalar fields.

[19] Taking $\kappa^2 = 1$ and $\eta_{RS} = diag\,(\,1_n,\ -1_6\,)$ as the $SO(n,6)$ metric.

7.1 The superpotential

Gauging supergravity in general leads to a Lagrangian with a scalar potential and mass terms for the gravitinos. In the $N = 4$ case, these term read $-(1/2)\,\mathcal{M}_{3/2}{}^{ij}\,\overline{\psi}_{\mu i}\sigma^{\mu\nu}\psi_{\nu j}+$ h.c., with $\mathcal{M}_{3/2}{}^{ij} = \mathcal{M}_{3/2}{}^{ji}$. In terms of the antisymmetric gauging structure constants f_{RST} and of the associated duality phases δ_R, as defined by de Roo and Wagemans [20, 21] (see section 5), the mass matrix is

$$\mathcal{M}_{3/2}{}^{ij} = -\frac{4}{3}\,\varphi^*_{(R)}\,f_{RST}\,\phi^{ikR}\phi^S_{kl}\phi^{ljT}\,, \tag{79}$$

and

$$\varphi^*_{(R)} = \sqrt{\frac{2}{S+\overline{S}}}\,\left(\cos\delta_R - iS\sin\delta_R\right)\,. \tag{80}$$

As indicated above, to obtain the $N = 1$ gravitino mass term, we formally reduce $SU(4)$ to $SU(3)$, splitting indices according to $i = (A, 4)$, $A = 1, 2, 3$, and we select $\psi_{\mu 4}$ as the $N = 1$ gravitino (*i.e.* we take $\psi_{\mu A} = 0$ in expression (79):

$$\mathcal{M}_{3/2}{}^{44} = -\frac{4}{3}\,\varphi^*_{(R)}\,f_{RST}\,\epsilon_{ABC}\phi^{A\,R}\phi^{B\,S}\phi^{C\,T}\,. \tag{81}$$

After replacing the $N = 4$ scalars by the solutions (77) of the Poincaré constraints truncated to $N = 1$, the holomorphic $N = 1$ superpotential W is obtained by equating this expression with

$$m_{3/2} = e^{K/2}\,W. \tag{82}$$

Separating the holomorphic and the real contributions leads to the Kähler potential

$$K = -\ln(S+\overline{S}) - \sum_{A=1}^{3}\,\ln Y(T_A, U_A, Z^I_A)\,, \tag{83}$$

while the superpotential is simply

$$W = \frac{4}{3}\sqrt{2}\,\left[\cos\delta_R - i\sin\delta_R S\right]\left[\prod_{A=1}^{3} Y(T_A, U_A, Z^I_A)\right]^{1/2} f_{RST}\,\epsilon_{ABC}\,\phi^{R\,A}\phi^{S\,B}\phi^{T\,C}\,. \tag{84}$$

It is a holomorphic function of (S, T_A, U_A, Z^I_A), once the $N = 4$ scalars from the vector multiplets have been truncated to $N = 1$ and replaced by the solutions (77).

Discarding the Z^I_A fields, the generic superpotential is a polynomial in the moduli fields with maximal degree seven. In particular, each monomial is of order zero or one in each of the seven moduli S, T_A, U_A. The superpotential can then have up to $2^7 = 128$ real parameters, which are structure constants and duality phases of the underlying $N = 4$ algebra[20]. These numbers can be identified with various fluxes of compactified string theories.

[20] The $N = 1$ truncation of the scalar fields $\phi^{R\,A}$ associates to each fixed value of $A = 1, 2, 3$ only four values of the index R, the four directions in each of the three $SO(2, 2)$. Hence f_{RST} includes $4^3 = 64$ real numbers.

Eqs. (83), (84) and (77) define completely the $N = 1$ effective supergravity with parameters f_{RST} (gauging structure constants) and δ_R (duality phases), up to consistency conditions on these parameters, as outlined in section 5.

8 String and supergravity moduli

To relate the construction of the effective four-dimensional supergravity described in the previous sections with string compactifications and fluxes, we need first to consider the $Z_2 \times Z_2$ orbifold reduction of the closed string sector, at the level of the fundamental geometric moduli states. In this context, the massless spectrum (before fluxes/gaugings are turned on) includes a string dilaton (*string coupling field*) φ, six metric moduli for the six internal radii, and their seven supersymmetric partner, for a total of fourteen scalar states corresponding to the supergravity fields S, T_A, U_A. The $Z_2 \times Z_2$ orbifold projection splits the six-dimensional internal space into three complex directions,

$$ds^2 = \sum_{A=1}^{3} ds_A^2,$$

each complex plane having a 2×2 metric $g_{A\,ab}$: $ds_A^2 = \sum_{a,b=1}^{2} g_{A\,ab}\, dx^a dx^b$. In complex coordinates,

$$ds_A^2 = \frac{1}{4}\, dz_A^2\, [g_{A\,11} - g_{A\,22} - 2ig_{A\,12}] + \frac{1}{4}\, d\overline{z}_A^2 [g_{A\,11} - g_{A\,22} + 2ig_{A\,12}] + \frac{1}{2}\, dz_A\, d\overline{z}_A [g_{A\,11} + g_{A\,22}],$$

and we define as usual

$$g_A = \frac{\hat{t}_A}{\hat{u}_A} \begin{pmatrix} \hat{u}_A^2 + \hat{\nu}_A^2 & \hat{\nu}_A \\ \hat{\nu}_A & 1 \end{pmatrix}, \qquad (\det g_A = \hat{t}_A^2). \tag{85}$$

In orbifolds Z_n, $n > 2$, the absence of dz_A^2 contributions leads to $g_{A\,11} = g_{A\,22}$ and $g_{A\,12} = 0$, or $\hat{u}_A = 1$ and $\hat{\nu}_A = 0$.

In the case of $Z_2 \times Z_2$, the metric modes $\hat{t}_A$ and $\hat{u}_A$ are always present. If not eliminated by the orientifold projection, the fields $\hat{\nu}_A$ assemble with $\hat{u}_A$ in $N = 1$ chiral supermultiplets with complex scalars

$$\hat{U}_A = \hat{u}_A + i\hat{\nu}_A. \tag{86}$$

The supersymmetry partners of the moduli $\hat{t}_A$ must be found among the massless modes of the heterotic ot type II NS–NS or R–R antisymmetric tensors. And two more massless states give rise to the supergravity dilaton S.

Dimensional reduction of the ten-dimensional Einstein term in the string frame on the metric

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}, \qquad g_{ij} = \begin{pmatrix} g_{A=1\,ab} & 0 & 0 \\ 0 & g_{A=2\,ab} & 0 \\ 0 & 0 & g_{A=3\,ab} \end{pmatrix}$$

clearly leads to a four-dimensional Einstein term with a dilaton and modulus-dependent prefactor:

$$-\frac{1}{2\kappa_{10}^2}\, e^{-2\varphi}\, e_{10}R_{10} \quad \longrightarrow \quad -\frac{1}{2\kappa_4^2} e^{-2\varphi}\,(\hat{t}_1\hat{t}_2\hat{t}_3)\, e_4 R_4.$$

Rescaling the vierbein according to

$$g_{\mu\nu} = e^{2\varphi}(\hat{t}_1\hat{t}_2\hat{t}_3)^{-1}\, \tilde{g}_{\mu\nu} \tag{87}$$

leads to the Einstein frame with gravitational Lagrangian $-\frac{1}{2\kappa_4^2}\, e_4 R_4$.

In each string compactification, it is essential to correctly identify the four-dimensional supergravity fields S, T_A, U_A in terms of the string massless modes. With this identification, the dependence on string moduli of, for instance, flux-induced potential contributions can be translated into a dependence of the effective superpotential on the chiral superfields S, T_A and U_A. It allows then, using the generic superpotential (84), to associate a given flux number with the corresponding gauging structure constant and then to completely translate the data of the ten-dimensional configuration into a certain gauging of the effective supergravity.

The appropriate identification strongly depends on the type of string theory (heterotic or type II) under consideration. It also depends on the orientifold projection used with type II strings. To conclude these notes, we briefly discuss this question, starting with the simplest and familiar case of the heterotic strings.

8.1 Heterotic strings

Heterotic gauge kinetic terms lead in four dimensions to

$$-\frac{1}{4}\, e^{-2\varphi}\, e_{10}F_{\mu\nu}F^{\mu\nu} \quad \longrightarrow \quad -\frac{1}{4}\, e^{-2\varphi}\,(\hat{t}_1\hat{t}_2\hat{t}_3)\, e_4 F_{\mu\nu}F^{\mu\nu} \tag{88}$$

both in Einstein and string frames. The natural definition of the real part of the chiral multiplet S is then

$$\mathrm{Re}\, S = e^{-2\varphi}\, \hat{t}_1\hat{t}_2\hat{t}_3 \equiv \hat{s}\,. \tag{89}$$

The supersymmetry partners of s, $\hat{t}_1$, $\hat{t}_2$ and $\hat{t}_3$ are the components of the two-form field[21] $B_{\mu\nu}$, B_{56}, B_{78} and B_{910}. Their kinetic terms derive from the three-form closed

[21] We use the simpler notation where $(5,6,7,8,9,10)$ replaces respectively $(A=1, a=1; A=1, a=2; A=2, a=1; A=2, a=2; A=3, a=1; A=3, a=2)$ for the six internal directions.

string Lagrangian term proportional to

$$\begin{aligned} e^{-2\varphi}\, e_{10}\, g^{MN} g^{PQ} g^{RS} H_{MPR} H_{NQS} \quad \longrightarrow \quad & e^{-2\varphi}\, e_4(\hat{t}_1\hat{t}_2\hat{t}_3) g^{\mu\nu} g^{\rho\sigma} g^{\lambda\tau} H_{\mu\rho\lambda} H_{\nu\sigma\tau}\,, \\ & e^{-2\varphi}\, e_4(\hat{t}_1\hat{t}_2\hat{t}_3) g^{\mu\nu} (\hat{t}_1)^{-2} H_{\mu 56} H_{\nu 56}\,, \\ & e^{-2\varphi}\, e_4(\hat{t}_1\hat{t}_2\hat{t}_3) g^{\mu\nu} (\hat{t}_2)^{-2} H_{\mu 78} H_{\nu 78}\,, \\ & e^{-2\varphi}\, e_4(\hat{t}_1\hat{t}_2\hat{t}_3) g^{\mu\nu} (\hat{t}_3)^{-2} H_{\mu 910} H_{\nu 910}\,. \end{aligned}$$

To simplify, this has been done assuming $\hat{\nu}_A = 0$. In the Einstein frame, after rescaling, these four-dimensional kinetic terms become

$$\begin{aligned} & e^{-4\varphi}\, e_4(\hat{t}_1\hat{t}_2\hat{t}_3)^2 g^{\mu\nu} g^{\rho\sigma} g^{\lambda\tau} H_{\mu\rho\lambda} H_{\nu\sigma\tau}, \\ & e_4\, (\hat{t}_1)^{-2}\, g^{\mu\nu} H_{\mu 56} H_{\nu 56}\,, \\ & e_4\, (\hat{t}_2)^{-2}\, g^{\mu\nu} H_{\mu 78} H_{\nu 78}\,, \\ & e_4\, (\hat{t}_3)^{-2}\, g^{\mu\nu} H_{\mu 910} H_{\nu 910}\,. \end{aligned}$$

These are the kinetic terms expected from Kähler potential

$$K = -\ln(S + \overline{S}) - \sum_{A=1}^{3} \ln(T_A + \overline{T}_A),$$

with the identification

$$\mathrm{Re}\, T_A = \hat{t}_A \tag{90}$$

and with the chiral S dualized to the dilaton linear superfield L which includes $B_{\mu\nu}$ and the real scalar

$$C \sim \hat{s}^{-1} \sim e^{2\varphi}\, (\hat{t}_1\hat{t}_2\hat{t}_3)^{-1}. \tag{91}$$

Isolating the dilaton dynamics corresponds to the choice $T_A = U_A = 1$. Hence, the heterotic dilaton theory is described by the chiral superfield S with Kähler potential $K = -\ln(S + \overline{S})$. In the language of $N = 1$ conformal supergravity with chiral compensating multiplet S_0 (with unit weights), the Lagrangian is

$$-\frac{3}{2}\Big[S_0\overline{S}_0\, e^{-K/3}\Big]_D = -\frac{3}{2}\Big[S_0\overline{S}_0\, (S + \overline{S})^{1/3}\Big]_D. \tag{92}$$

The dual version with linear multiplet L is [28]

$$-\Big[(S_0\overline{S}_0)^{3/2} L^{-1/2}\Big]_D. \tag{93}$$

With these identifications, one easily obtains for instance that the superpotential generated by fluxes of the three-form field depends on U_A. Consider for instance the component H_{689}, which is allowed by the $Z_2 \times Z_2$ projection. After rescaling to the

Einstein frame, the Lagrangian term quadratic in $H_{(3)}$ leads to a contribution to the scalar potential of the form

$$\sim e_4\, e^K\, \hat{u}_1^2\hat{u}_2^2\, H_{689}H_{689}\,,$$

where $e^{-K} \propto \hat{s}\prod_A(\hat{t}_A\hat{u}_A)$. The effective $N=1$ superpotential includes then a contribution $W \sim H_{689}\, U_1U_2$. Eight components of $H_{(3)}$ survive the $Z_2\times Z_2$ projection. They correspond to superpotential terms proportional to $1, U_A, U_{[A}U_{B]}$ and $U_1U_2U_3$. It is a nontrivial test of consistency to verify that the complexification required by $N=1$ supergravity is already present in the ten-dimensional theory. The relation with the gauging structure constants is finally established by replacing the $N=1$ chiral fields by $N=4$ constrained fields. In our example,

$$\begin{aligned} &U_A = i\sqrt{Y}\,(\rho_A^2-\sigma_A^2), \qquad 1 = \sqrt{Y}\,(\rho_A^1+\sigma_A^1) \\ &\qquad\longrightarrow \quad W \sim -Y^{3/2}\, H_{689}\,(\rho_1^2-\sigma_1^2)(\rho_2^2-\sigma_2^2)(\rho_3^1+\sigma_3^1). \end{aligned} \tag{94}$$

Comparison with the general expression (84) of the superpotential indicates which f_{RST} correespond to the flux H_{689}. In this example, all duality phases vanish and the superpotential does not depend on S: this is of course a general property of heterotic moduli superpotentials.

Eqs. (89) and (85) define the real fields $\hat{s}, \hat{t}_A$ and $\hat{u}_A$ as a function of the string dilaton φ and the metric radius modes (in the string frame). These definitions will apply as well to the type II cases discussed below. It is however only true for heterotic strings that $\mathrm{Re}\, S = \hat{s}$, $\mathrm{Re}\, T_A = \hat{t}_A$ and $\mathrm{Re}\, U_A = \hat{u}_A$, where S, T_A, U_A are the four-dimensional chiral fields with Kähler potential (83).

8.2 Type IIB strings, orientifold with D9 (and D5) branes

In this orientifold, gauge fields live on D9 and D5 branes and the massless component of the four-dimensional antisymmetric tensor arises from the R–R field $C_{(2)}$. Its kinetic Lagrangian is

$$\sim e_4[e^{\varphi}\,(\hat{t}_1\hat{t}_2\hat{t}_3)^{-1}]^{-2}\, F_{(3)\mu\nu\rho}F_{(3)}{}^{\mu\nu\rho}, \tag{95}$$

after the vierbein rescaling to the Einstein frame. This suggests that $C_{(2)\mu\nu}$ belongs to a linear multiplet with real scalar component

$$C = e^{\varphi}\,(\hat{t}_1\hat{t}_2\hat{t}_3)^{-1}. \tag{96}$$

The kinetic term of a gauge field living on a D9–brane is

$$-\frac{1}{4}e_4\, e^{-\varphi}\,(\hat{t}_1\hat{t}_2\hat{t}_3)\, F_{\mu\nu}F^{\mu\nu}, \tag{97}$$

while for a gauge field living on a D5 one obtains

$$-\frac{1}{4}e_4\, e^{-\varphi}\,\hat{t}_A\, F_{\mu\nu}F^{\mu\nu}. \tag{98}$$

These results hold in the string and Einstein frames. These gauge kinetic terms define the chiral fields $s = \mathrm{Re}\, S \sim C^{-1}$ and $t_A = \mathrm{Re}\, T_A$ in terms of the string dilaton and metric modes, according to

$$\begin{aligned} s &= e^{-\varphi}\,\hat{t}_1\hat{t}_2\hat{t}_3 = \sqrt{\hat{s}\hat{t}_1\hat{t}_2\hat{t}_3}\,, & t_1 &= e^{-\varphi}\,\hat{t}_1 = \sqrt{\hat{s}\hat{t}_1/\hat{t}_2\hat{t}_3}\,, \\ t_2 &= e^{-\varphi}\,\hat{t}_2 = \sqrt{\hat{s}\hat{t}_2/\hat{t}_1\hat{t}_3}\,, & t_3 &= e^{-\varphi}\,\hat{t}_3 = \sqrt{\hat{s}\hat{t}_3/\hat{t}_1\hat{t}_2}\,. \end{aligned} \tag{99}$$

The Kähler potential is again eq. (83).

The massless components $C_{(2)56}$, $C_{(2)78}$ and $C_{(2)910}$ have kinetic terms

$$\begin{aligned} &\sim e_4\, e^{2\varphi}\,\hat{t}_1^{-2}\, g^{\mu\nu} F_{(3)\mu 56}F_{(3)\nu 56}\,, \\ &\sim e_4\, e^{2\varphi}\,\hat{t}_2^{-2}\, g^{\mu\nu} F_{(3)\mu 78}F_{(3)\nu 78}\,, \\ &\sim e_4\, e^{2\varphi}\,\hat{t}_3^{-2}\, g^{\mu\nu} F_{(3)\mu 910}F_{(3)\nu 910} \end{aligned} \tag{100}$$

in the Einstein frame. This is as predicted by Kähler potential (83), with identification $\mathrm{Im}\, T_1 \sim C_{(2)56}$, $\mathrm{Im}\, T_2 \sim C_{(2)78}$ and $\mathrm{Im}\, T_3 \sim C_{(2)910}$. The U_A fields are as in heterotic strings.

8.3 Type IIB strings, orientifold with D3 (and D7) branes

The kinetic term of gauge fields living respectively on a D3 or a D7 branes are of the form

$$-\frac{1}{4}e_4\, e^{-\varphi}\, F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad -\frac{1}{4}e_4\, e^{-\varphi}\,(\hat{t}_1\hat{t}_2 \quad \text{or} \quad \hat{t}_2\hat{t}_3 \quad \text{or} \quad \hat{t}_3\hat{t}_1)\, F_{\mu\nu}F^{\mu\nu}. \tag{101}$$

There are four massless modes of the R–R tensors $C_{(0)}$ and $C_{(4)}$, with kinetic terms

$$\begin{aligned} &\sim e_4\, e^{2\varphi}\,(\partial_\mu C)(\partial^\mu C), & &\sim e_4\, e^{2\varphi}\,(\hat{t}_1\hat{t}_2)^{-2}\,(\partial_\mu C_{5678})(\partial^\mu C_{5678}), \\ &\sim e_4\, e^{2\varphi}\,(\hat{t}_1\hat{t}_3)^{-2}\,(\partial_\mu C_{56910})(\partial^\mu C_{56910}), & &\sim e_4\, e^{2\varphi}\,(\hat{t}_2\hat{t}_3)^{-2}\,(\partial_\mu C_{78910})(\partial^\mu C_{78910}). \end{aligned} \tag{102}$$

The identifications of the $N = 1$ chiral fields $\mathrm{Re}\, S = s$ and $\mathrm{Re}\, T_A = t_A$ in terms of the string dilaton and metric modes are then:

$$\begin{aligned} s &= e^{-\varphi} = \sqrt{\hat{s}/\hat{t}_1\hat{t}_2\hat{t}_3}\,, & t_1 &= e^{-\varphi}\,\hat{t}_2\hat{t}_3 = \sqrt{\hat{s}\hat{t}_2\hat{t}_3/\hat{t}_1}\,, \\ t_2 &= e^{-\varphi}\,\hat{t}_3\hat{t}_1 = \sqrt{\hat{s}\hat{t}_3\hat{t}_1/\hat{t}_1}\,, & t_3 &= e^{-\varphi}\,\hat{t}_1\hat{t}_2 = \sqrt{\hat{s}\hat{t}_1\hat{t}_2/\hat{t}_3}\,. \end{aligned} \tag{103}$$

The Kähler potential is again eq. (83) and $ImS \sim C$, $\mathrm{Im}\, T_1 \sim C_{78910}$, $\mathrm{Im}\, T_2 \sim C_{56910}$ and $\mathrm{Im}\, T_3 \sim C_{5678}$. The U_A fields are as in heterotic strings.

8.4 Type IIA strings, orientifold with D6 branes

The orientifold acts on the internal space-time coordinates according to $z_A \to -\overline{z}_A$. Coordinates x^5, x^7 and x^9 are odd. The metric modes ν_A are then projected out.

The fourteen massless moduli are the string dilaton, the six diagonal modes $\hat{t}_A$ and $\hat{u}_A$, four R–R states C_{6810}, C_{679}, C_{589} and C_{5710} and three NS–NS states B_{56}, B_{78} and B_{910}. The kinetic terms indicate that $\mathrm{Re}\, T_A = \hat{t}_a$ as in heterotic strings and that the scalars T_A get complexified using the NS–NS states.

Gauge field on D6–branes have kinetic terms

$$-\frac{1}{4} e_4\, e^{-\varphi} \sqrt{\hat{t}_1 \hat{t}_2 \hat{t}_3 / \hat{u}_1 \hat{u}_2 \hat{u}_3}\, F_{\mu\nu} F^{\mu\nu}. \tag{104}$$

The gauge coupling define $\mathrm{Re}\, S = s$, and $\mathrm{Im}\, S \propto C_{6810}$.

Finally, the expressions of the R–R kinetic terms suggest the identifications of the $N = 1$ chiral fields $\mathrm{Re}\, S = s$ and $\mathrm{Re}\, U_A = u_A$ in terms of the string dilaton and metric modes:

$$\begin{array}{rcccl} s & = & e^{-\varphi} \sqrt{\hat{t}_1 \hat{t}_2 \hat{t}_3 / \hat{u}_1 \hat{u}_2 \hat{u}_3} & = & \sqrt{\hat{s} / \hat{u}_1 \hat{u}_2 \hat{u}_3}\,, \\ u_1 & = & e^{-\varphi} \sqrt{\hat{t}_1 \hat{t}_2 \hat{t}_3 \hat{u}_2 \hat{u}_3 / \hat{u}_1} & = & \sqrt{\hat{s} \hat{u}_2 \hat{u}_3 / \hat{u}_1}\,, \\ u_2 & = & e^{-\varphi} \sqrt{\hat{t}_1 \hat{t}_2 \hat{t}_3 \hat{u}_1 \hat{u}_3 / \hat{u}_2} & = & \sqrt{\hat{s} \hat{u}_1 \hat{u}_3 / \hat{u}_2}\,, \\ u_3 & = & e^{-\varphi} \sqrt{\hat{t}_1 \hat{t}_2 \hat{t}_3 \hat{u}_1 \hat{u}_2 / \hat{u}_3} & = & \sqrt{\hat{s} \hat{u}_1 \hat{u}_2 / \hat{u}_3}\,. \end{array} \tag{105}$$

The Kähler potential is again eq. (83).

It is interesting to remark that in type II strings, the dilaton dynamics is governed by the Kähler potential

$$K_{type\ II} = -4 \ln(S + \overline{S}). \tag{106}$$

As in the heterotic string, the dynamics of the string dilaton φ is isolated from moduli couplings by choosing $\hat{t}_A = \hat{u}_A = 1$ [*i.e.* assuming unit value for all six radii in the string frame (85)]. In type IIB strings, this leads to $S = T_1 = T_2 = T_3$, $U_A = 1$. In the type IIA orientifold, this corresponds to $S = U_1 = U_2 = U_3$ and $T_A = 1$. And in both cases, the Kähler potential is given by eq. (106). This Kähler potential, which differs from the heterotic one, reflects the particular expansion in powers of the string coupling field which characterizes open strings or type II strings and orientifolds.

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